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Existence of solutions of elliptic boundary value problems with mixed type nonlinearities

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Abstract
We study the existence of a nontrivial solution of the following elliptic boundary value problem with mixed type nonlinearities:

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(x,u) = -K_u + W_u$. We consider the problem in a different case: $\lim_{|u|\to\infty} f(x,u)/u = \infty$, $\lim_{|u|\to 0} f(x,u)/u$ is some constant. Assuming that K satisfies the "pinching" condition, and W satisfies a more general superquadratic growth condition than the well-known Ambrosetti-Rabinowitz condition usually used in literature, we obtain a nontrivial solution via the Mountain Pass Lemma.
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Keywords: pinching condition; Mountain Pass Lemma; Cerami condition

1 Introduction

In this paper, we shall be concerned with the elliptic boundary value problem in a different case

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset R^N$ $(N > 2)$ is a bounded open domain with a smooth boundary $\partial\Omega$ and $f \in C(\Omega \times R^1, R^1)$.

The existence of nontrivial weak solutions for (P) have been studied in many papers, see [1–12]. Su and Zhao in [2] considered problem (P) for resonance case at infinity, $\lim_{|u|\to\infty} \frac{f(x,u)}{u} = \lambda_k$, where λ_k is an eigenvalue of the linear boundary value problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_0}$$

the existence of multiple nontrivial solutions for (P) are obtained by minimax methods and Morse theory. Ambrosetti and Rabinowitz in [3] established the existence of a nontrivial solution for problem (P) by assuming the following conditions:

(f_1') $f(x,0)=0$, $\lim_{u\to 0}\frac{f(x,u)}{u}=0$, uniformly in a.e. $x\in\Omega$.
(f_2') There exist two positive constants a and b such that

$$\left|f(x,u)\right|\le a+b|u|^p \quad \text{for some } 0\le p<\frac{N+2}{N-2}, \forall u\in R^1, x\in\Omega.$$

And the following well-known Ambrosetti-Rabinowitz condition ((AR) for short):

$$\exists\theta>2, R_0>0 \quad \text{s.t. } 0<\theta F(x,u)\le uf(x,u), \quad \text{for all } |u|\ge R_0, x\in\Omega,$$

where $F(x,u)=\int_0^u f(x,s)\,ds$.

Since then, the (AR) condition has been used extensively in many literature sources, see [12–18]. It is well known that the (AR) condition is quite natural and convenient not only to ensure that the Euler-Lagrange functional associated to problem (P) has a mountain pass geometry but also to guarantee that the Palais-Smale sequence of the Euler-Lagrange functional is bounded. Let E be a Hilbert space and $G\in C^1(E,R^1)$. Recall that the sequence $\{u_n\}_{n\in N}\subset E$ is said to be a Palais-Smale sequence of G provided that $\{G(u_n)\}$ is bounded and $G'(u_n)\to 0$ as $n\to+\infty$, the function G satisfies the Palais-Smale condition ((PS) for short) if and only if any Palais-Smale sequence for G contains a convergent subsequence. The function G satisfies the Cerami condition ((C) for short) if any sequence $\{u_n\}_{n\in N}$ in E satisfying $G(u_n)$ is bounded and $G'(u_n)(1+\|u_n\|)\to 0$ as $n\to+\infty$ has a convergent subsequence.

Without (AR), it becomes more complicated. Indeed, there are many functions which are superlinear, but it is not necessary to satisfy (AR) even if $1<\theta\le 2$. Willem and Zou stated the following examples:

$$f(x,u)=\mu|u|^{\mu-2}u+(\mu-1)|u|^{\mu-3}u\sin^2 u+|u|^{\mu-1}\sin 2u, \quad u\in R^1\setminus\{0\},$$

where $\mu>2$. Then it is easy to check that (AR) does not hold even for any $\theta>\mu-1>1$. On the other hand, in order to verify (AR), it usually is an annoying task to compute a primitive function of f and sometimes it is almost impossible. For example,

$$f(x,u)=|u|u\left(1+e^{(1+|\sin u|)^\alpha}+|\cos u|^\alpha\right), \quad u\in R^1,$$

where $\alpha>0$.

Some authors have tried to drop or weaken the above superlinear condition (AR) in recent years, see [4–8, 11, 19]. Miyagaki and Souto [8] adapted some monotonicity arguments studying the existence of nontrivial weak solutions of (P).

The aim of the manuscript is to consider the problem in a different case: $\lim_{|u|\to\infty}f(x,u)/u=\infty$, $\lim_{|u|\to 0}f(x,u)/u$ is some constant. We study this problem under "pinching" condition and the general superquadratic condition. The case that $F(x,u)$ has a part with "pinching" condition has been considered only by few authors, see [13, 20]. Since $F(x,u)$ does not satisfy the (f') and (AR), problem (P) becomes more delicate. The main difficulty when dealing with this problem is the lack of compactness of the Sobolev embedding theorem.

In this paper, here, $F(t,u) := \int_0^u f(t,s)\,ds$ replaced by $-K + W$, satisfy

(F_1) $F(x,u) = -K(x,u) + W(x,u)$, $K, W : \Omega \times R^1 \to R^1$ are C^1-maps.

(K_1) There are two positive constants b_1 and b_2 such that

$$b_1|u|^2 \le K(x,u) \le b_2|u|^2, \quad \text{for all } (x,u) \in \Omega \times R^1.$$

(K_2) There exists $\varrho \in (1,2]$ such that

$$K(x,u) \le K_u(x,u)u \le \varrho K(x,u), \quad \text{for all } (x,u) \in \Omega \times R^1.$$

(W_1) $W(x,u) \ge 0$ and $W_u(x,u) = o(|u|)$ as $|u| \to 0$ uniformly in x.

(W_2) $W(x,u)/u^2 \to \infty$ as $|u| \to \infty$ uniformly in x.

(W_3) Set $\tilde{W}(x,u) := \frac{1}{2}W_u(x,u)u - W(x,u)$, $\tilde{W}(x,u) > 0$ if $u \neq 0$, $\tilde{W}(x,u) \to \infty$ as $|u| \to \infty$ uniformly in x, and there exist $r_0 > 0$ and $\sigma > N/2$ such that $|W_u(x,u)|^\sigma \le c_0\tilde{W}(x,u)|u|^\sigma$ if $|u| \ge r_0$.

We will prove the following results.

Theorem 1.1 *If assumptions (F_1), (K_1), (K_2) and (W_1)-(W_3) are satisfied, then problem (P) has a nontrivial weak solution.*

Remark 1

(i) Our assumptions (W_2), (W_3) are weaker than (AR), and there is no monotone condition;

(ii) The condition (K_2) can be written in the form $1 \le \frac{K_u(x,u)u}{K(x,u)} \le \varrho$, $\varrho \in (1,2]$ which is weaker than the condition $1 \le \frac{K_u(x,u)u}{K(x,u)} \le 2$ in [13, 20].

Example 1 Consider the functions

$$K(x,u) = \bigl[1 + \exp\bigl(-|x|\bigr)\bigr]u^2, \qquad W(x,u) = \left(2 - \frac{1}{1+|x|}\right)u^2 \ln\bigl(1+u^2\bigr).$$

A straightforward computation shows that K and $W(x,u)$ satisfy the assumptions of Theorem 1.1, but neither $F(x,u)$ nor $W(x,u)$ satisfy the (AR) condition.

Example 2 Consider the more general functions

$$K_u(x,u) = V(x)u, \qquad W_u(x,u) = g(x,u),$$

where $g(x,u)$ is of superlinear growth as $|u| \to \infty$. A straightforward computation shows that K and W satisfy the assumptions of Theorem 1.1.

We will prove that the function associated with (P) has Mountain Pass geometry and satisfies the (C) condition. The remainder of the paper is organized as follows. In Section 2, we deal with the variational setting. In Section 3, we give the details of the proof of Theorem 1.1.

2 Preliminary results

Let $H := H_0^1(\Omega)$ be the Sobolev space equipped with the inner product and the norm

$$(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\,dx, \qquad \|u\| = (u,u)^{\frac{1}{2}}, \quad u,v \in H.$$

And we denote the usual $L^p(\Omega)$-norm

$$\|u\|_p = \left(\int_\Omega \left|u(x)\right|^p dx\right)^{\frac{1}{p}}.$$

Our approach will be the variational techniques. Define the Euler-Lagrange functional associated to problem (P) given by

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega \left[-K(x,u) + W(x,u)\right]dx, \quad \text{for all } u \in H.$$

From the assumptions on f, it is standard to check that $\Phi \in C^1$ whose Gateaux derivative is

$$\Phi'(u)v = \int_\Omega \nabla u \cdot \nabla v\,dx - \int_\Omega \left[-K_u(x,u)v + W_u(x,u)v\right]dx, \quad \text{for all } u,v \in H.$$

Let $\eta : H \to [0,+\infty)$ be given by

$$\eta(u) := \left(\int_\Omega \left[|\nabla u|^2 + 2K(x,u)\right]dx\right)^{\frac{1}{2}}.$$

Hence

$$\Phi(u) = \frac{1}{2}\eta^2(u) - \int_\Omega W(x,u)\,dx.$$

By (K_2),

$$\Phi'(u)u \le \int_\Omega |\nabla u|^2\,dx + \int_\Omega \varrho K(x,u)\,dx - \int_\Omega W_u(x,u)u\,dx.$$

By (K_1) and set $b_{1,1} := \min\{1, 2b_1\}$, $b_{2,2} := \max\{2, 2b_2\}$,

$$b_{1,1}\|u\|^2 \le \eta(u)^2 \le b_{2,2}\|u\|^2.$$

It is worth pointing out that if the function $K(x,u)$ is of the form $\frac{1}{2}V(x)u^2$ with $V(x) \in C^1(\Omega, R^1)$ and $\inf_\Omega V(x) \ge V_0 > 0$ then η in a Hilbert space $X = \{u \in H_0^1(\Omega); \int_\Omega V(x)u^2 < \infty\}$ is equivalent to the norm $\|\cdot\|$; however, if the function $K(x,u)$ is not of the form $\frac{1}{2}V(x)u^2$, η is not a norm because of the lack of norm's linear property.

Lemma 2.1 (see [5]) *Let H be a real Banach space, $\Phi \in C^1(H,R)$, satisfying $\Phi(0) = 0$. Moreover,*

(i) there exist $\rho, \alpha > 0$ such that $\Phi_{|\partial B_\rho(0)} \ge \alpha$,

(ii) there exists $e \in H \setminus \overline{B_\rho(0)}$ such that $\Phi(e) \leq 0$.
Then there exists a sequence $\{u_n\} \in H$ such that $\|\Phi'(u_n)\|(1+\|u_n\|) \to 0$ and $\Phi(u_n) \to c \geq \alpha$ as $n \to \infty$.

Lemma 2.2 (see [1]) *Assume that $|\Omega| < \infty$, $1 \leq p$, $r < \infty$, $f \in C(\bar{\Omega} \times R)$ and $|f(x,u)| \leq c(1 + |u|^{\frac{p}{r}})$. Then, for every $u \in L^p(\Omega)$, $f(x,u) \in L^r(\Omega)$ and the operator $A : L^p(\Omega) \to L^r(\Omega)$, $u \mapsto f(x,u)$ is continuous.*

3 Proofs of theorems

First of all, we recall a property of the function $K(x,u)$, which is necessary to the proof of the geometric structure of the C^1 functional Φ.

Fact 1 *Assume that (K_2) holds, then*

$$K(x,u) \leq K\left(x, \frac{u}{|u|}\right)|u|^{\varrho}, \quad \text{for all } x \in \Omega \text{ and } |u| \geq 1.$$

Proof Define $G : s \to K(x, s^{-1}u)s^{\varrho}$, $s \in (0, +\infty)$

$$\begin{aligned} G'(s) &= -K_u\big(x, s^{-1}u\big)\frac{u}{s^2}s^{\varrho} + K\big(x, s^{-1}u\big)\varrho s^{\varrho-1} \\ &= -K_u\big(x, s^{-1}u\big)s^{-1}us^{\varrho-1} + K\big(x, s^{-1}u\big)\varrho s^{\varrho-1} \\ &= s^{\varrho-1}\big[-K_u\big(x, s^{-1}u\big)s^{-1}u + K\big(x, s^{-1}u\big)\varrho\big]. \end{aligned}$$

By (K_2), $G'(s) \geq 0$, which implies $G(s)$ is non-decreasing. So, we have

$$K(x,u) = G(1) \leq G(s) = K\left(x, \frac{u}{|u|}\right)|u|^{\varrho}, \quad \text{if } |u| = s \geq 1.$$

Next we discuss the geometric structure of the C^1 functional Φ on H. □

Lemma 3.1 *Under the assumptions of Theorem 1.1, there are constants $\rho, \alpha > 0$ such that $\Phi_{|\partial B_\rho(0)} \geq \alpha$.*

Proof From (W_1) and (W_3), as $|u| > r_0$, we have

$$\begin{aligned} \big|W_u(x,u)\big|^{\sigma} &\leq c_0\left(\frac{1}{2}W_u(x,u)u - W(x,u)\right)|u|^{\sigma} \\ &\leq c_1 W_u(x,u)|u|^{\sigma+1}, \end{aligned}$$

where c_1 is a positive constant. Hence as $|u| > r_0$,

$$\big|W_u(x,u)\big| \leq c_1|u|^{\sigma+1/(\sigma-1)}.$$

From $\sigma > N/2$, we know $\sigma + 1/(\sigma - 1) < 2^* - 1$, so we can choose $2\sigma/(\sigma - 1) \leq p < 2^*$. And using (W_1) again, we observe that for any given $\epsilon > 0$ there is $c_\epsilon > 0$ such that

$$\big|W_u(x,u)\big| \leq \epsilon|u| + c_\epsilon|u|^{p-1} \tag{3.1}$$

and

$$\left|W(x,u)\right| \le \epsilon|u|^2 + c_\epsilon|u|^p. \tag{3.2}$$

It follows from (3.2) and the Sobolev embedding theorem that for all $u \in H$

$$\int_\Omega W(x,u)\,dx \le \epsilon\|u\|_2^2 + c_\epsilon\|u\|_p^p \le \epsilon\|u\|_2^2 + c_\epsilon c\|u\|^p, \tag{3.3}$$

where c is a positive constant. Then combining (K_1) and (3.3), we obtain

$$\begin{aligned}\Phi(u) &\ge \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + b_1\|u\|_2^2 - \left(\epsilon\|u\|_2^2 + c_\epsilon c\|u\|^p\right)\\ &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + (b_1-\epsilon)\|u\|_2^2 - c_\epsilon c\|u\|^p \ge \min\left\{\frac{1}{2},(b_1-\epsilon)\right\}\|u\|^2 - c_\epsilon c\|u\|^p\end{aligned}$$

set $b_0 = min\{\frac{1}{2},(b_1-\epsilon)\}$, it is clear that $b_0 > 0$. We choose $\|u\| = \rho = (\frac{\frac{1}{2}b_0}{c_\epsilon c})^{\frac{1}{p-2}}$ and $\alpha = \frac{1}{2}\rho^2 b_0$, then

$$\Phi(u) \ge \alpha.$$

□

Lemma 3.2 *Under the assumptions of Theorem 1.1, there exists $e \in H \setminus \overline{B_\rho(0)}$ such that $\Phi(e) < 0$.*

Proof Let $e_0 \in H \setminus 0$, $M = \max_{x\in\Omega,|u|\le 1} K(x,u)$ and $A > \frac{(2M+1)\|e_0\|^2}{2\|e_0\|_2^2}$. By (W_2), there exists $B > 0$ such that

$$W(x,u) \ge A|u|^2 - B, \quad \text{for all } x \in \Omega, u \in H. \tag{3.4}$$

As $\varrho < 2$, by Fact 1, we have

$$\begin{aligned}\eta^2(\xi e_0) &= \int_\Omega \left[|\nabla \xi e_0|^2 + 2K(x,\xi e_0)\right]dx\\ &\le \int_\Omega |\nabla e_0|^2\xi^2\,dx + 2\int_{\{x\in\Omega;|\xi e_0|\le 1\}} K(x,\xi e_0)\,dx + 2\int_{\{x\in\Omega;|\xi e_0|\ge 1\}} K(x,\xi e_0)\,dx\\ &\le \int_\Omega |\nabla e_0|^2\xi^2\,dx + 2M|\Omega| + 2M\int_{\{x\in\Omega;|\xi e_0|\ge 1\}} |\xi e_0|^\varrho\,dx\\ &\le \int_\Omega |\nabla e_0|^2\xi^2\,dx + 2M|\Omega| + 2M\int_{\{x\in\Omega;|\xi e_0|\ge 1\}} |\xi e_0|^2\,dx\\ &\le \xi^2(1+2M)\|e_0\|^2 + 2M|\Omega|.\end{aligned} \tag{3.5}$$

Then, by inequalities (3.4) and (3.5), we get

$$\begin{aligned}\Phi(\xi e_o) &= \frac{1}{2}\eta^2(\xi e_0) - \int_\Omega W(x,\xi e_0)\,dx \le \frac{1+2M}{2}\xi^2\|e_0\|^2 + M|\Omega| - A\xi^2\|e_0\|_2^2 - B|\Omega|\\ &= \left(\frac{1+2M}{2}\|e_0\|^2 - A\|e_0\|_2^2\right)\xi^2 + (M-B)|\Omega|.\end{aligned} \tag{3.6}$$

By the choice of A, we have $(\frac{1+2M}{2}\|e_0\|^2 - A\|e_0\|_2^2) < 0$, so there exists $\xi_0 \in R^1$ such that if $e = \xi_0 e_0$, then

$$\Phi(e) < 0. \qquad \square$$

Suppose that the assumptions of Theorem 1.1 hold, we have Lemma 3.1 and Lemma 3.2. Now it follows from Lemma 2.1 that there is a sequence $\{u_n\} \subset H$ such that

$$\|\Phi'(u_n)\|\bigl(1 + \|u_n\|\bigr) \to 0 \quad \text{and} \quad \Phi(u_n) \to c \geq \alpha \quad \text{as } n \to \infty. \tag{3.7}$$

Lemma 3.3 *Under the assumptions of Theorem 1.1, the functional Φ satisfies the (C) condition.*

Proof Let $\{u_n\} \subset H$ be such that

$$\Phi(u_n) \text{ is bounded} \quad \text{and} \quad \bigl(1 + \|u_n\|\bigr)\Phi'(u_n) \to 0. \tag{3.8}$$

By (K_2) we observe that for large n,

$$\begin{aligned} c_2 &\geq \Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n \\ &= \int_\Omega K(x,u_n)\,dx - \int_\Omega \frac{1}{2}K_u(x,u_n)u_n\,dx + \int_\Omega \frac{1}{2}W_u(x,u_n)u_n - W(x,u_n)\,dx \\ &\geq \int_\Omega K(x,u_n) - \frac{\varrho}{2}K(x,u_n)\,dx + \int_\Omega \frac{1}{2}W_u(x,u_n)u_n - W(x,u_n)\,dx \\ &\geq \int_\Omega \tilde{W}(x,u_n)\,dx. \end{aligned} \tag{3.9}$$

Arguing indirectly, assume as a contradiction that $\|u_n\| \to \infty$. Setting $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$ and since the embedding $H \hookrightarrow L^s$ for $s \in [2, 2^*)$, we have $\|v_n\|_s \leq \gamma_s\|v_n\| = \gamma_s$. Observe that, from (3.8), (K_1) and (K_2)

$$\begin{aligned} \Phi'(u_n)u_n &= \int_\Omega |\nabla u_n|^2\,dx + \int_\Omega K_u(x,u_n)u_n\,dx - \int_\Omega W_u(x,u_n)u_n\,dx \\ &\geq \int_\Omega |\nabla u_n|^2\,dx + \int_\Omega K(x,u_n)\,dx - \int_\Omega W_u(x,u_n)u_n\,dx \\ &\geq \int_\Omega |\nabla u_n|^2\,dx + \int_\Omega b_1|u_n|^2\,dx - \int_\Omega W_u(x,u_n)u_n\,dx \\ &\geq \int_\Omega \frac{1}{2}|\nabla u_n|^2\,dx + \int_\Omega b_1|u_n|^2\,dx - \int_\Omega W_u(x,u_n)u_n\,dx \\ &\geq \|u_n\|^2\left(\frac{1}{2}b_{1,1} - \int_\Omega \frac{W_u(x,u_n)v_n}{\|u_n\|}\,dx\right). \end{aligned}$$

It follows that for any $\epsilon > 0$ and n large enough,

$$\int_\Omega \frac{W_u(x,u_n)v_n}{\|u_n\|}\,dx \geq \frac{b_{1,1}}{2} - \epsilon. \tag{3.10}$$

Set for $r \geq 0$

$$h(r) := \inf\{\tilde{W}(x,u) : x \in \Omega \text{ and } u \in R^1 \text{ with } |u| \geq r\}.$$

By (W_3), $h(r) > 0$ for all $r > 0$ and $h(r) \to \infty$ as $r \to \infty$. For $0 \leq a < b$, let

$$\Omega_n(a,b) = \{x \in \Omega : a \leq |u_n(x)| < b\}$$

and

$$C_a^b = \inf\left\{\frac{\tilde{W}(x,u)}{u^2}\middle| x \in \Omega \text{ and } u \in R^1 \text{ with } a \leq |u_n(x)| < b\right\}.$$

Since $\tilde{W}(x,u) > 0$ if $u \neq 0$, one has $C_a^b > 0$ and

$$\tilde{W}(x, u_n(x)) \geq C_a^b |u_n(x)|^2 \quad \text{for all } x \in \Omega_n(a,b).$$

It follows from (3.9) that

$$\begin{aligned} c_2 &\geq \int_\Omega \tilde{W}(x,u_n)\,dx \\ &= \int_{\Omega_n(0,a)} \tilde{W}(x,u_n)\,dx + \int_{\Omega_n(a,b)} \tilde{W}(x,u_n)\,dx + \int_{\Omega_n(b,\infty)} \tilde{W}(x,u_n)\,dx \\ &\geq \int_{\Omega_n(0,a)} \tilde{W}(x,u_n)\,dx + C_a^b \int_{\Omega_n(a,b)} |u_n(x)|^2\,dx + h(b)|\Omega_n(b,\infty)|. \end{aligned} \tag{3.11}$$

Set $\tau := 2\sigma/(\sigma-1)$, since $\sigma > N/2$, one sees $\tau \in (2, 2^*)$. Fix arbitrarily $\hat{\tau} \in (\tau, 2^*)$, using (3.11),

$$|\Omega_n(b,\infty)| \leq \frac{c_2}{h(b)} \to 0 \quad \text{as } b \to \infty \text{ uniformly in } n,$$

which implies by the Hölder inequality that

$$\begin{aligned} \int_{\Omega_n(b,\infty)} |v_n|^\tau\,dx &\leq \left(\int_{\Omega_n(b,\infty)} 1\,dx\right)^{1-\frac{\tau}{\hat{\tau}}} \left(\int_{\Omega_n(b,\infty)} |v_n|^{\tau\frac{\hat{\tau}}{\tau}}\,dx\right)^{\frac{\tau}{\hat{\tau}}} \\ &\leq \gamma_{\hat{\tau}}^{\tau} |\Omega_n(b,\infty)|^{1-\frac{\tau}{\hat{\tau}}} \\ &\to 0 \end{aligned} \tag{3.12}$$

as $b \to \infty$ uniformly in n. Using (3.11) again, for any fixed $0 < a < b$,

$$\int_{\Omega_n(a,b)} |v_n|^2\,dx = \frac{1}{\|u_n\|^2} \int_{\Omega_n(a,b)} |u_n|^2\,dx \leq \frac{c_2}{C_a^b \|u_n\|^2} \to 0$$

as $n \to \infty$. Let $0 < \epsilon < \frac{b_{1,1}}{8}$, by (W_1), there exists $a_\epsilon > 0$ such that

$$|W_u(x,u)| \leq \frac{\epsilon}{\gamma_2}|u|, \quad \text{for all } |u| \leq a_\epsilon.$$

Consequently,

$$\int_{\Omega_n(0,a_\epsilon)} \frac{W_u(x,u_n)}{|u_n|}|v_n|^2\,dx \le \int_{\Omega_n(0,a_\epsilon)} \frac{\epsilon}{\gamma_2}|v_n|^2\,dx \le \frac{\epsilon}{\gamma_2}\|v_n\|_2^2 \le \epsilon \tag{3.13}$$

for all n. By (W_3) and (3.12), we can take large $b_\epsilon \ge r_0$ so that

$$\begin{aligned}\int_{\Omega_n(b_\epsilon,\infty)} \frac{W_u(x,u_n)}{|u_n|}|v_n|^2\,dx &\le \left(\int_{\Omega_n(b_\epsilon,\infty)} \frac{|W_u(x,u_n)|^\sigma}{|u_n|^\sigma}\,dx\right)^{\frac{1}{\sigma}} \left(\int_{\Omega_n(b_\epsilon,\infty)} |v_n|^{2\sigma'}\,dx\right)^{\frac{2}{2\sigma'}}\\ &\le \left(\int_\Omega c_0\tilde{W}(x,u_n)\,dx\right)^{\frac{1}{\sigma}} \left(\int_{\Omega_n(b_\epsilon,\infty)} |v_n|^\tau\,dx\right)^{\frac{2}{\tau}}. \end{aligned}\tag{3.14}$$

Hence combining (3.11), (3.12) and (3.14), there is n_0 such that

$$\int_{\Omega_n(b_\epsilon,\infty)} \frac{|W_u(x,u_n)||v_n|^2}{|u_n|}\,dx \le (c_2c_0)^{\frac{1}{\sigma}} \left(\int_{\Omega_n(b_\epsilon,\infty)} |v_n|^\tau\,dx\right)^{\frac{2}{\tau}} < \epsilon \tag{3.15}$$

for $n \ge n_0$. Note that there is $\gamma = \gamma(\epsilon) > 0$ independent of n such that

$$\big|W_u(x,u_n)\big| \le \gamma|u_n| \quad \text{for } x \in \Omega_n(a_\epsilon,b_\epsilon).$$

By (3.12)

$$\int_{\Omega_n(a_\epsilon,b_\epsilon)} \frac{|W_u(x,u_n)||v_n|^2}{|u_n|}\,dx \le \gamma\int_{\Omega_n(a_\epsilon,b_\epsilon)} |v_n|^2\,dx < \epsilon \tag{3.16}$$

for all $n \ge n_0$. Therefore, combining (3.14)-(3.16), we obtain for $n \ge n_0$

$$\int_\Omega \frac{|W_u(x,u_n)||v_n|^2}{|u_n|}\,dx \le 3\epsilon < \frac{b_{1,1}}{2} - \epsilon$$

which contradicts (3.10). Hence $\{u_n\}$ is bounded in H. Going if necessary to a subsequence, we assume that

$$u_n \rightharpoonup u \quad \text{in } H \text{ for some } u \in H,$$

which implies $u_n \to u$ a.e. in Ω, because the imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Hence we have $\|u_n - u\|_2 \to 0$ and $|(\Phi'(u_n) - \Phi'(u))(u_n - u)| \to 0$. Using the Hölder inequality

$$\begin{aligned}&\left|\int_\Omega \big(f\big(x,u_n(x)\big) - f\big(x,u(x)\big)\big)\big(u_n(x) - u(x)\big)\,dx\right|\\ &\quad\le \left(\int_\Omega \big|f\big(x,u_n(x)\big) - f\big(x,u(x)\big)\big|^q\right)^{\frac{1}{q}} \left(\int_\Omega \big|u_n(x) - u(x)\big|^p\right)^{\frac{1}{p}}\end{aligned}$$

$(\frac{1}{p} + \frac{1}{q} = 1)$ for $u_n \to u$ in $L^p(\Omega)$, and by (K_1), (K_2), (W_1), (W_3) we have

$$\big|f(x,u)\big| \le c\big(1 + |u|^{p-1}\big) = c\big(1 + |u|^{\frac{p}{q}}\big).$$

Then, by Lemma 2.2, we have $f(x, u_n(x)) \to f(x, u(x))$ in $L^q(\Omega)$. Thus

$$\int_\Omega \bigl(f\bigl(x, u_n(x)\bigr) - f\bigl(x, u(x)\bigr)\bigr)\bigl(u_n(x) - u(x)\bigr)\,dx \to 0$$

as $n \to +\infty$. Moreover, a straightforward computation shows that

$$\bigl(\Phi'(u_n) - \Phi'(u)\bigr)(u_n - u) = \bigl\|\nabla(u_n - u)\bigr\|_2^2 - \int_\Omega \bigl(f\bigl(x, u_n(x)\bigr) - f\bigl(x, u(x)\bigr)\bigr)\bigl(u_n(x) - u(x)\bigr)\,dx$$

it is clear that

$$\bigl\|\nabla(u_n - u)\bigr\|_2^2 \to 0. \tag{3.17}$$

Finally,

$$\bigl\|u_n - u\bigr\| \to 0 \quad \text{in } H.$$

This completes the proof. □

Now, we are ready to prove Theorem 1.1.

We will obtain a critical point of Φ_λ by the use of a standard version of the Mountain Pass Lemma (see [3]). It provides the minimax characterization for the critical value which is important for what follows. Therefore, we state this lemma precisely.

Lemma 3.4 (see [3]) *Let H be a real Banach space and $\Phi_\lambda : H \to R^1$ be a C^1-smooth functional. If Φ satisfies the following conditions:*

(i) $\Phi(0) = 0$,

(ii) every sequence $\{u_n\}_{n\in N}$ in H such that $\{\Phi(u_n)\}_{n\in N}$ is bounded in R^1 and $\Phi'(u_n) \to 0$ in H^* as $n \to +\infty$, contains a convergent subsequence ((PS) condition),

(iii) there are constants $\rho, \alpha > 0$ such that $\Phi_{|\partial B_\rho(0)} \geq \alpha$,

(iv) there is a constant $e \in H \setminus \overline{B_\rho(0)}$ such that $\Phi(e) \leq 0$,

where $B_\rho(0)$ is an open ball in H of radius ρ centered at 0, then Φ possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} \Phi\bigl(g(s)\bigr),$$

where

$$\Gamma = \bigl\{g \in C\bigl([0,1], H\bigr)\bigr\} : g(0) = 0, g(1) = e\}.$$

Now we are ready to give the proofs of Theorem 1.1.

Proof of Theorem 1.1 Under conditions (F_1), (K_1), (K_2), (W_1)-(W_3), as shown in [9], a deformation lemma can be proved with the (C) condition, replacing the usual Palais-Smale condition, and it turns out that the Mountain Pass Theorem still holds true. Applying the Mountain Pass Lemma 3.4, Φ possesses a critical value $c \geq \alpha$ given by $c =$

$\inf_{g\in\Gamma} \max_{s\in[0,1]} \Phi(g(s))$. Hence, u is a nontrivial solution of problem (P) satisfying $\Phi(u) = c$, $\Phi'(u) = 0$. The proof is done. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The paper is the result of joint work of all authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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